Indian Statistical Institute

Midterm Examination 2022-2023

Analysis of Several Variables, B.Math Second Year

Time : 3 Hours   Date : 19.09.2022   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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Q1. (15 marks) Suppose $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^m \to \mathbb{R}^p$ are linear maps. What is the derivative of BA? Justify your answer.

Q2. (15 marks) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be C^2-functions. Suppose

$$F(x, y) = f(x + y) + g(x - y) \qquad (x, y \in \mathbb{R}).$$

Then $\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} =?$

Q3. (15 marks) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that f is partially differentiable everywhere, and determine where f is differentiable.

Q4. (15 marks) Find and then classify all critical points of

$$f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x \qquad ((x, y) \in \mathbb{R}^2).$$

Q5. (15 marks) Let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open convex subset, and let $f : \mathcal{O} \to \mathbb{R}^m$ be a differentiable function. If $(Df)(x) = 0$ for all $x \in \mathcal{O}$, then prove that f is a constant function.

Q6. (15 marks) Let $\mathcal{O} \subseteq \mathbb{R}^2$ be an open subset, and let $f : \mathcal{O} \to \mathbb{R}$ be a continuous function of x (for each fixed y). Suppose $\frac{\partial f}{\partial y}$ is a bounded function on $\mathcal{O}$. Prove that f is continuous.

Q7. (20 marks) Let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open convex subset, and let $f : \mathcal{O} \to \mathbb{R}^m$ be a differentiable function. Suppose that a and b are distinct points in $\mathcal{O}$. Prove that there exists a point c on the line segment joining a and b such that

$$\|f(a) - f(b)\| \leq \|Df(c)(a - b)\|.$$